known journalist, the founder of the Chicago *Daily News*, and the general manager of the Associated Press.

Professor Stone has written various papers on mathematical and astronomical subjects, which have appeared from time to time in the *Astronomische Nachrichten*, in *Gould's Astronomical Journal*, and in the *Annals of Mathematics*.

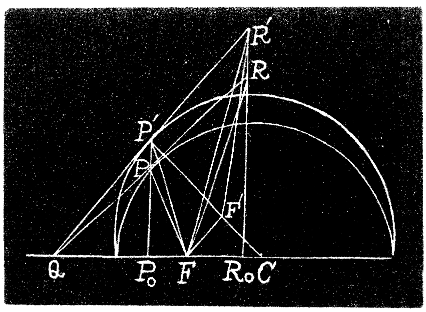
Professor Stone is also a member of a number of learned societies. In 1888 he was Chairman of the section of Mathematics and Astronomy of the American Association for the Advancement of Science ; and he is at present a member of the Council of the American Mathematical Society.

AN ELEMENTARY DERIVATION OF THE LAW OF GRAVITATION AS APPLIED TO PLANETARY MOTIONS.

By ORMOND STONE, University of Virginia.

The following derivation of the law of gravitation from Kepler's first two laws of planetary motion without the use of the machinery of the infinitesimal calculus is a modification of that given by Moebius. The loss by fire of a large portion of the library of the University of Virginia prevents my giving the place in his works where it may be found. As given by Moebius a slight knowledge of solid geometry is required ; as here given all the operations are performed in the plane of the orbit. The mass of the planet has been neglected.

Draw a circle having the major axis of the orbit as a diameter. Assume a point P' having such a motion that it is always at the intersection of the circumference of this circle and a straight line drawn through the planet P perpendicular to the major axis of the planet's orbit. The components of the velocities of P and P' in the direction parallel to the major axis are thus equal.



Draw QR tangent to the ellipse at P, and QR' tangent to the circle at P'. Q is situated on the major axis extended. If $PR=V$ represent the velocity of P and $P'R'=V'$ represent the velocity of P', RR' will be parallel to PP'. Let P_0 and R_0 be the intersections of PP' and RR' with the major axis of the orbit. Then by a property of the ellipse

$$P_0P=P_0P'\cos\varphi,\ R_0R=R_0R'\cos\varphi,$$

in which φ is the angle whose sine is e, the eccentricity of the orbit.

By one of Kepler's laws the sun is at F, the focus of the ellipse. PRF represents the areal velocity of P, and $P'R'F$ the areal velocity of P' with reference to F. As is easily seen,

$$PRF = P'R'F \cos \varphi ;$$

whence, since by one of Kepler's laws PRF is constant, $P'R'F = c'$ is also a constant, and the acceleration of P' is directed toward F (see Young's General Astronomy, Art. 406).

Let A and A' be the total accelerations of P and P', and A_0 and A_0' be the components of these accelerations parallel to the major axis of the ellipse. Evidently

$$\frac{A_0}{A} = \frac{P_0F}{PF}, \quad \frac{A_0'}{A'} = \frac{P_0F}{P'F};$$

whence, since $A_0 = A_0'$,

$$\frac{A}{A'} = \frac{PF}{P'F}. \tag{1}$$

Let C be the center of the ellipse, and F'' the foot of the perpendicular from F on $P'C$. The component of A' in the direction $P'C$ is

$$A' \cos FP'F'' = A' \frac{F''P'}{FP'} = \frac{V'^2}{a} \tag{2}$$

(see Young's General Astronomy, Art. 411).

Put $\angle FCP' = E =$ eccentric anomaly, and $FP = r =$ radius vector. We have also

$$\begin{aligned} CF &= ae, \\ CF'' &= ae \cos E, \\ F''P' = CP' - CF'' &= a(1 - e \cos E), \\ P_0P &= a \cos \varphi \sin E, \\ FP_0 &= a(\cos E - e). \end{aligned}$$

The last two equations give

$$FP = \sqrt{\overline{FP_0}^2 + \overline{P_0P}^2} = a(1 - e \cos E);$$

whence

$$F''P' = FP = r,$$

and (1) and (2) give

$$A = A' \frac{F''P'}{FP'} = \frac{V'^2}{a}. \tag{3}$$

Since FF' is parallel to $P'R'$, the area of $P'F'R'$ is equal to that of $P'FR'$, which has already been shown to be constant; whence

$$F'P' \times P'R' = rV' = 2c', \text{ or } V' = \frac{2c'}{r}.$$

Substituting this in (3), we have

$$A = \frac{4c'^2}{a} \cdot \frac{1}{r^2}. \qquad \text{Q. E. D.}$$

A NOTE ON MEAN VALUES.

By E. H. MOORE, Ph. D., Professor of Mathematics in the University of Chicago.

A problem in averages or mean values usually reads thus:

(A) *Given a certain totality $\Omega[\psi]$ of objects ψ, and a certain function $f(\psi)$ of every object ψ; required the mean value f_Ω of the $f(\psi)$ for the ψ's of the totality $\Omega[\psi]$.*

If the totality $\Omega[\psi]$ contains a *finite* number n of objects ψ — $\psi_1, \psi_2 \ldots . \psi_n$ — then we have the formula

$$(1) \qquad f_\Omega = \frac{\sum_{i=1}^{n} f(\psi_i)}{n}.$$

If the totality $\Omega[\psi]$ does not contain a finite number of objects, then the problem as stated (A) is *indefinite*. [The solution (1) cannot be directly generalized. To say that the number n is ∞ means merely that the totality $\Omega[\psi]$ is without number, that there is *no* such *number n*.]

To make (A) definite we must supplement it by an explicit statement of a law of distribution of the objects ψ over the totality $\Omega[\psi]$. In the ordinary cases this law of distribution makes ψ depend uniquely upon certain m independent variables $u_1, u_2 \ldots . u_m$, write it $\psi = \psi(u_1 \ldots . u_m)$, in such a way that the totality $\Omega[\psi]$ defines a certain totality $\bar{\Omega}[u_1 \ldots . u_m]$, and the function $f(\psi)$ becomes $f(\psi) = \bar{f}(u_1 \ldots . u_m)$. Now if the m-ple definite integrals

$$(2) \qquad I_1 = \int \ldots . \int \bar{f}(u_1 \ldots . u_m)\, du_1 \ldots . du_m, \quad I_2 = \int \ldots . \int du_1 \ldots . du_m$$

taken over the totality $\bar{\Omega}[u_1 \ldots . u_m]$ have definite meaning, (whether *or not* the